Math 241

Problem Set 4 solution manual

Exercise. A4.1

First recall that $D_n = \{1, r, r^2, ..., r^{n-1}, s, rs, ..., r^{n-1}s\}$, and $A_n = \{\sigma \in S_n \mid sign(\sigma) = +1\}$. So our goal is to check which of the elements of D_n have a sign of $+1$.

We know that $r = (123...n)$, so sign of $r = \begin{cases} -1 & n \ even \\ +1 & n \ odd \end{cases}$,

and $s = \begin{cases} for \ \ n = 4k & s = (1 \ \ n)(2 \ \ n-1)...(2k \ \ 2k+1) \\ for \ \ n = 4k+2 & s = (1 \ \ n)(2 \ \ n-1)...(2k+1 \ \ 2k+2) \\ for \ \ n = 4k+1 & s = (1 \ \ n)(2 \ \ n-1)...(2k \ \ 2k+2) \\ for \ \ n = 4k+3 & s = (1 \ \ n)(2 \ \ n-1)...(2k+1 \ \ 2k+3) \end{cases}$

So we conclude that sign(s)=$\begin{cases} for \ \ n = 4k & sign(s) = +1 \\ for \ \ n = 4k+2 & sign(s) = -1 \\ for \ \ n = 4k+1 & sign(s) = +1 \\ for \ \ n = 4k+3 & sign(s) = -1 \end{cases}$

Then we can know the elements of $A_n \cap D_n$ according to the value of n, and we conclude that:

$A_n \cap D_n$=$\begin{cases} for \ \ n = 4k & A_n \cap D_n = \{1, r^2, r^4, .., r^{n-2}, s, r^2 s, ..., r^{n-2}s\} \\ for \ \ n = 4k+2 & A_n \cap D_n = \{1, r^2, r^4, .., r^{n-2}, rs, r^3 s, ..., r^{n-1}s\} \\ for \ \ n = 4k+1 & A_n \cap D_n = \{1, r, r^2, .., r^{n-1}, s, rs, ..., r^{n-1}s\} = D_n \\ for \ \ n = 4k+3 & A_n \cap D_n = \{1, r, r^2, .., r^{n-1}\} \end{cases}$

Exercise. A4.2

a- Required to prove that $detA = detA^t$

First we show that $sign(\sigma) = sign(\sigma^{-1})$, to prove this we need to notice that $\sigma.\sigma^{-1} = 1$ and $sign(\sigma) = +1$, then $sign(\sigma).sign(\sigma^{-1}) = 1$, which implies that $sign(\sigma) = sign(\sigma^{-1})$.

Now back to the problem:

$detA = \sum_{\sigma \in S_n} sign(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} ... a_{n\sigma(n)}$

Now notice that for every i we have $\sigma(i) = j$ for some unique $j \implies a_{i\sigma(i)} = a_{\sigma^{-1}(j)j}$, and this is true for all $1 \leq i \leq n$ then we can deduce that $a_{1\sigma(1)} a_{2\sigma(2)} ... a_{n\sigma(n)} = a_{\sigma^{-1}(1)1} a_{\sigma^{-1}(2)2} ... a_{\sigma^{-1}(n)n}$.

And since $\{\sigma \mid \sigma \in S_n\} = \{\sigma^{-1} \mid \sigma \in S_n\}$.

We deduce that:

$detA = \sum_{\sigma^{-1} \in S_n} sign(\sigma^{-1}) a_{\sigma^{-1}(1)1} a_{\sigma^{-1}(2)2} ... a_{\sigma^{-1}(n)n}$

$= \sum_{\sigma^{-1} \in S_n} sign(\sigma^{-1}) a_{\sigma^{-1}(1)1} a_{\sigma^{-1}(2)2} ... a_{\sigma^{-1}(n)n}$

$= \sum_{\sigma \in S_n} sign(\sigma) a_{\sigma(1)1} a_{\sigma(2)2} ... a_{\sigma(n)n}$

Now let $A^t = (b_{ij})$ with $b_{ij} = a_{ji}$, then:
$detA^t = \sum\limits_{\sigma \in S_n} sign(\sigma) b_{1\sigma(1)} b_{2\sigma(2)} ... b_{n\sigma(n)}$
$= \sum\limits_{\sigma \in S_n} sign(\sigma) a_{\sigma(1)1} a_{\sigma(2)2} ... a_{\sigma(n)n}$
$= detA.$

b- A is an upper triangular $\implies a_{ij}$=0 when evere $i > j$.

But for evey $\sigma \neq 1 \in S_n \ \exists \ i$ such that $i > \sigma(i)$.

Then let $\sigma \neq 1$ we will get $a_{1\sigma(1)} a_{2\sigma(2)} ... a_{n\sigma(n)} = 0$ since it contains at least one element $a_{i\sigma(i)}$ with $i > \sigma(i)$ which implies $a_{i\sigma(i)} = 0$.

Then the only term that appears in the sum over $\sigma \in S_n$ is for $\sigma = 1 \implies detA = sign(1) a_{1\sigma(1)} a_{2\sigma(2)} ... a_{n\sigma(n)}$ with $\sigma = 1$, $\implies detA = a_{11} a_{22} ... a_{nn}$.

c- Consider that the two interchanged columns are the q^{th}, and the p^{th}, and consider $\tau = (p \ q)$, and let $B = (b_{ij})$ be the matrix A after we interchange the two columns, and note that $a_{ij} = b_{ij}$ for all $j \notin \{p, q\}$, and $a_{ip} = b_{iq}$, $a_{iq} = b_{ip}$

let $H = \{1, \tau\}$ then we get $\{\tau o \sigma \mid \sigma \in S_n\} = \bigcup\limits_{\sigma \in S_n} \sigma H = \{\sigma \mid \sigma \in S_n\}$ since the set of cosets of H forms a partition of S_n.

Then $detB = \sum\limits_{\sigma \in S_n} sign(\sigma) b_{1\sigma(1)} b_{2\sigma(2)} ... b_{n\sigma(n)}$
$= \sum\limits_{\tau o \sigma \in S_n} sign(\tau o \sigma) b_{1\tau o \sigma(1)} b_{2\tau o \sigma(2)} ... b_{n\tau o \sigma(n)}$
$= \sum\limits_{\tau o \sigma \in S_n} sign(\tau) sign(\sigma) b_{1\tau o \sigma(1)} b_{2\tau o \sigma(2)} ... b_{n\tau o \sigma(n)}$
$= -\sum\limits_{\tau o \sigma \in S_n} sign(\sigma) b_{1\tau o \sigma(1)} b_{2\tau o \sigma(2)} ... b_{n\tau o \sigma(n)}$

Choose any $\sigma \in S_n$, and suppose $\sigma(i) = p$, and $\sigma(j) = q$ for some $i, j \in \{1, ..., n\}$)

$$\text{Then } \begin{cases} for \ \ s \neq i, \ \ s \neq j & we \ \ have \ \ b_{s\tau o \sigma(s)} = b_{s\sigma(s)} = a_{s\sigma(s)}. \\ for \ \ s = i, & we \ \ have \ \ b_{i\tau o \sigma(i)} = b_{i\tau(p)} = b_{iq} = aip = a_{i\sigma(i)}. \\ for \ \ s = j, & we \ \ have \ \ b_{j\tau o \sigma(j)} = b_{j\tau(q)} = b_{jp} = ajq = a_{j\sigma(j)}. \end{cases}$$

Then

$detB = -\sum\limits_{\tau o \sigma \in S_n} sign(\sigma) b_{1\tau o \sigma(1)} b_{2\tau o \sigma(2)} ... b_{n\tau o \sigma(n)}$
$= -\sum\limits_{\tau o \sigma \in S_n} sign(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} ... a_{n\sigma(n)}$
$= -\sum\limits_{\sigma \in S_n} sign(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} ... a_{n\sigma(n)}$
$= -detA.$

d- Let A be such that A has two equal columns, then if we interchange those two columns of A we will get a matrix $A' = A$.

Moreover, $detA' = -detA$, but since $A = A'$ then also $detA' = detA$,
then we get $detA = -detA$, which implies $detA = 0$.

Section. 9

Exercise. 7

σ=(1 4 5)(7 8)(2 5 7)

- $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 1 & 3 & 5 & 8 & 6 & 2 & 7 \end{bmatrix}$
  The backward pairs : (1,2), (1,3), (1,7), (3,7), (4,7), (5,6), (5,7), (5,8), (6,7).
  So $N_\sigma = 9 \implies \text{sign}(\sigma) = (-1)^9 = -1$.

- σ=(1 5)(1 4)(7 8)(2 7)(2 5)
  $\implies \text{sign}(\sigma) = (-1)^5 = -1$

- σ=(1 4 5 8 7 2) so σ is a k-cycle with k=6
  $\implies \text{sign}(\sigma) = (-1)^5 = -1$.

Exercise. 10

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{bmatrix}$$

- The backward pairs : (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (2,8), (3,4), (3,6), (3,7), (3,8), (4,8), (5,6), (5,7), (5,8), (6,8), (7,8).
  So $N_\sigma = 18 \implies \text{sign}(\sigma) = (-1)^{18} = +1$.

- σ=(7 5)(3 4)(3 6)(1 8)
  $\implies \text{sign}(\sigma) = (-1)^4 = +1$

- σ=(1 8)(3 6 4)(5 7) so σ is a product of 3 disjoint cycles τ_1, τ_2, τ_3 with
  $\text{sign}(\tau_1) = -1$, $\text{sign}(\tau_2) = +1$, $\text{sign}(\tau_3) = -1$
  $\implies \text{sign}(\sigma) = (-1)(+1)(-1) = +1$.

Exercise. 31

A is an infinite set, H is a subset of S_A such that the elements of H moves only a finite number of elements.

- The identity element of S_A moves no elements of A, $\implies$ it moves a finite number of elements $\implies 1_A \in H$.

- Let σ_1, and $\sigma_2 \in H$, then since σ_1, and σ_2 moves only a finite number of elements of A then they can be written as the finite product of cycles of finite order, $\implies \sigma_1\sigma_2$ is the finite product of cycles of finite order, then $\sigma_1\sigma_2 \in H$.

- If σ moves only a finite number of element then write σ as the finite product of disjoint cycles of finite order, then σ has a finite order, let n =order of σ, this implies that $\sigma^{-1} = \sigma^{n-1}$ which is the finite product of cycles of finite order, $\implies \sigma^{-1} \in H$.

$\implies H$ is a subgroup of S_A.

Exercise. 32

We have K is a subset of S_A such that every element in K moves at most 50 elements of A.
Then consider σ_1=(1 2 ... 50) , and σ_2=(50 ... 60).
Then $\sigma_1\sigma_2$=(1 2 ... 50)(50 ... 60)= (1 2 ... 60) , which clearly moves 60 elements ,
$\implies \sigma_1\sigma_2 \notin H$.
So H is not closed under the operation of the group. $\implies H$ is not a subgroup of S_A.

Section. 10

Exercise. 2

The cosets of $4\mathbb{Z}$ in $2\mathbb{Z}$ are :
$a + 4\mathbb{Z}$ for any $a \in 2\mathbb{Z}$, but any $a \in 2\mathbb{Z}$, $a = 4k$, or $a = 4k + 2$
$\implies$ for $a = 4k$ we have one coset which is $4\mathbb{Z}$, and for $a = 4k+2$ we have another coset which is $2 + 4\mathbb{Z}$.

Exercise. 4

The cosets of $< 4 >$ in $\mathbb{Z}_{12}$ are :
$0+ < 4 >=< 4 >= \{0, 4, 8\} = 4+ < 4 >= 8+ < 4 >$.
$1+ < 4 >= \{1, 5, 9\} = 5+ < 4 >= 9+ < 4 >$.
$2+ < 4 >= \{2, 6, 10\} = 6+ < 4 >= 10+ < 4 >$.
$3+ < 4 >= \{3, 7, 11\} = 7+ < 4 >= 11+ < 4 >$.

Exercise. 6

The left cosets of $H = \{\rho_0, \mu_2\}$ in D_4 are:
$\rho_0.H = \mu_2.H = \{\rho_0, \mu_2\}$.
$\rho_1.H = \delta_2.H = \{\rho_1, \delta_2\}$.
$\rho_2.H = \mu_1.H = \{\rho_2, \mu_1\}$.
$\rho_3.H = \delta_1.H = \{\rho_3, \delta_1\}$.

Exercise. 7

The right cosets of $H = \{\rho_0, \mu_2\}$ in D_4 are:
$H.\rho_0 = H.\mu_2 = \{\rho_0, \mu_2\}$.
$H.\rho_1 = H.\delta_1 = \{\rho_1, \delta_1\}$.
$H.\rho_2 = H.\mu_1 = \{\rho_2, \mu_1\}$.
$H.\rho_3 = H.\delta_2 = \{\rho_3, \delta_2\}$.
So the left cosets are not equal to the right cosets.

Exercise. 8

	ρ_0	μ_2	ρ_1	δ_2	ρ_2	μ_1	ρ_3	δ_1
ρ_0	ρ_0	μ_2	ρ_1	δ_2	ρ_2	μ_1	ρ_3	δ_1
μ_2	μ_2	ρ_0	δ_1	ρ_3	μ_1	ρ_2	δ_2	ρ_1
ρ_1	ρ_1	δ_2	ρ_2	μ_1	ρ_3	δ_1	ρ_0	μ_2
δ_2	δ_2	ρ_1	μ_2	ρ_0	δ_1	ρ_3	μ_1	ρ_2
ρ_2	ρ_2	μ_1	ρ_3	δ_1	ρ_0	μ_2	ρ_1	δ_2
μ_1	μ_1	ρ_2	δ_2	ρ_1	μ_2	ρ_0	δ_1	ρ_3
ρ_3	ρ_3	δ_1	ρ_0	μ_2	ρ_1	δ_2	ρ_2	μ_1
δ_1	δ_1	ρ_3	μ_1	ρ_2	δ_2	ρ_1	μ_2	ρ_0

So we can deduce form the table directly that we don't have a coset group.

Exercise. 9

The left cosets of $H = \{\rho_0, \rho_2\}$ in D_4 are:
$\rho_0.H = \rho_2.H = \{\rho_0, \rho_2\}$.
$\rho_1.H = \rho_3.H = \{\rho_1, \rho_3\}$.
$\mu_1.H = \mu_2.H = \{\mu_1, \mu_2\}$.
$\delta_1.H = \delta_2.H = \{\delta_1, \delta_2\}$.

Exercise. 10

The right cosets of $H = \{\rho_0, \rho_2\}$ in D_4 are:
$H.\rho_0 = H.\rho_2 = \{\rho_0, \rho_2\}$.
$H.\rho_1 = H.\rho_3 = \{\rho_1, \rho_3\}$.
$H.\mu_1 = H.\mu_2 = \{\mu_1, \mu_2\}$.
$H.\delta_1 = H.\delta_2 = \{\delta_1, \delta_2\}$.
So the left cosets are equal to the right cosets.

Exercise. 11

	ρ_0	ρ_2	ρ_1	ρ_3	μ_1	μ_2	δ_1	δ_2
ρ_0	ρ_0	ρ_2	ρ_1	ρ_3	μ_1	μ_2	δ_1	δ_2
ρ_2	ρ_2	ρ_0	ρ_3	ρ_1	μ_2	μ_1	δ_2	δ_1
ρ_1	ρ_1	ρ_3	ρ_2	ρ_0	δ_1	δ_2	μ_2	μ_1
ρ_3	ρ_3	ρ_1	ρ_0	ρ_2	δ_2	δ_1	μ_1	μ_2
μ_1	μ_1	μ_2	δ_2	δ_1	ρ_0	ρ_2	ρ_3	ρ_1
μ_2	μ_2	μ_1	δ_1	δ_2	ρ_2	ρ_0	ρ_1	ρ_3
δ_1	δ_1	δ_2	μ_1	μ_2	ρ_1	ρ_3	ρ_0	ρ_2
δ_2	δ_2	δ_1	μ_2	μ_1	ρ_3	ρ_1	ρ_2	ρ_0

And this is a coset group with the following table:

	red	blue	green	black
red	red	blue	green	black
blue	blue	red	black	green
green	green	black	red	blue
black	black	green	blue	red

And since for every x in this group we have x^2 =identity of the group (in this case "red"), this group is isomorphic to the klein-4 group V.

Exercise. 30

False, Here is a counter example:
Let $G = S_3$ and let $H = \{id, (23)\}$, $a = (123)$, $b = (12)$.
Then $a.H = b.H = \{(123)(12)\}$
While $H.a = \{(123)(13)\} \neq H.b$ since $H.b = \{(132)(12)\}$.

Exercise. 31

If $H.a = H.b$ Now since , $H.b = \{h.b \mid h \in H\}$ and since $e \in H$ then we get $b \in H.b = H.a$ $\implies b \in H.a.$

Exercise. 32
True, since:
We know that it is enough to show that there intersection is non-empty:
Now since $a.H = b.H \implies b \in a.H$ (similar to above argument) $\implies b = a.h$ for some $h \in H$ then $b^{-1} = h^{-1}.a^{-1}$
$\implies b^{-1} \in H.a^{-1}$, but $b^{-1} \in H.b^{-1} \implies H.a^{-1} = H.b^{-1}$

Exercise. 33

False, and here is a counter example:
Let $G = D_4$, and let $H = \{\rho_0, \mu_2\}$.
Let $a = \rho_1$ then $a^2 = \rho_2$, and $b = \delta_2$ then $b^2 = \rho_0$,
we have $a.H = b.H = \{\rho_1, \delta_2\}$, while $a^2.H = \{\rho_2, \mu_1$ different form $b^2.H = \{\rho_0, \mu_2\}$.